A UNIVERSAL PROBABILITY DISTRIBUTION FUNCTION
FOR WEAK-LENSING AMPLIFICATION

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ABSTRACT

We present an approximate form for the weak lensing magnification distribution of standard candles, valid for all cosmological models, with arbitrary matter distributions, over all redshifts. Our results are based on a universal probability distribution function (UPDF), $P(\eta)$, for the reduced convergence, η. For a given cosmological model, the magnification probability distribution, $P(\mu)$, at redshift z is related to the UPDF by $P(\mu) = P(\eta)/2\,|\kappa_{min}|$, where $\eta = 1 + (\mu - 1)/(2|\kappa_{min}|)$, and κ_{min} (the minimum convergence) can be directly computed from the cosmological parameters (Ω_m and Ω_Λ). We show that the UPDF can be well approximated by a three-parameter stretched Gaussian distribution, where the values of the three parameters depend only on ξ_η, the variance of η. In short, all possible weak lensing probability distributions can be well approximated by a one-parameter family. We establish this family, normalizing to the numerical ray-shooting results for a ΛCDM model by Wambsganss et al. (1997). Each alternative cosmological model is then described by a single function $\xi_\eta(z)$. We find that this method gives $P(\mu)$ in excellent agreement with numerical ray-tracing and three-dimensional shear matrix calculations, and provide numerical fits for three representative models (SCDM, ΛCDM, and OCDM). Our results provide an easy, accurate, and efficient method to calculate the weak lensing magnification distribution of standard candles, and should be useful in the analysis of future high-redshift supernova data.

Subject headings: cosmology: observations—cosmology: theory—gravitational lensing

1. INTRODUCTION

The luminosity distance-redshift relations of cosmological standard candles provide a powerful probe of the cosmological parameters H_0, Ω_m, and Ω_Λ (Garnavich et al. 1998a; Perlmutter et al. 1999; Wang 2000b; Branch et al. 2001), as well as of the nature of the dark energy (Garnavich et al. 1998b; White 1998; Podariu & Ratra 2000; Waga & Frieman 2000; Maor et al. 2001; Podariu, Nugent, & Ratra 2001; Wang & Garnavich 2001; Wang & Lovelace 2001; Kujat et al. 2002). At present, type Ia supernovae (SNe Ia) are our best candidates for cosmological standard candles (Phillips 1993; Riess, Press, & Kirshner 1995). The main systematic uncertainties of SNe Ia as cosmological standard candles are weak gravitational lensing (Kantowski et al. 1995; Frieman 1997; Wambsganss et al. 1997; Holz 1998; Holz & Wald 1998; Wang 1999; Valageas 2000a,b; Munshi & Jain 2000; Barber et al. 2000; Premadi et al. 2001), and luminosity evolution (Drell, Loredo, & Wasserman 2000; Riess et al. 1999; Wang 2000b). Future SN surveys (Wang 2000a, SNAP[1]) could yield thousands of SNe Ia out to redshifts of a few. Since the effect of weak lensing increases with redshift, the appropriate modeling of the weak lensing of high-redshift SNe Ia will be important in the correct interpretation of future data. In addition, with high statistics it may be possible to directly measure the lensing distributions, and thereby infer properties of the dark matter (Metcalf & Silk 1999; Seljak & Holz 1999).

In general, determining the magnification distributions of standard candles due to weak lensing is a laborious and time-consuming process, involving such techniques as ray-tracing through N-body simulations or Monte-Carlo approximations to inhomogeneous universes. Here we present an easy, accurate, and efficient method to calculate the weak lensing magnification distribution of standard candles, $P(\mu)$. Our method avails itself of a universal probability distribution function (UPDF), $P(\eta)$, which we fit to a simple analytic form (normalized by the cosmological N-body simulations of Wambsganss et al. (1997)). All weak lensing magnification probability distributions, for all cosmological models over all redshifts, can then be approximated by a one-parameter family of solutions. The underlying fundamental parameter is ξ_η, the variance of the reduced convergence, η. To determine the magnification PDF for a given model it is thus sufficient to determine $\xi_\eta(z)$ for that model. We demonstrate this method with a number of examples, and provide fitting formulae for three fiducial cosmologies (see Table 1).

Table 1
Three fiducial models

Model	Ω_m	Ω_Λ	h	σ_8
SCDM	1.0	0.0	0.5	0.6
ΛCDM	0.3	0.7	0.7	0.9
OCDM	0.3	0.0	0.7	0.85

2. WEAK LENSING OF POINT SOURCES

Due to the deflection of light by density fluctuations along the line of sight, a source (at redshift z_s) will be magnified by a factor $\mu \simeq 1 + 2\kappa$ (the weak lensing limit),

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[1] see http://snap.lbl.gov

where the convergence κ is given by (Bernardeau, Van Waerbeke, & Mellier 1997; Kaiser 1998)

$$\kappa = \frac{3}{2}\,\Omega_m \int_0^{\chi_s} \mathrm{d}\chi\, w(\chi,\chi_s)\,\delta(\chi), \qquad (1)$$

with

$$\mathrm{d}\chi = \frac{cH_0^{-1}\,dz}{\sqrt{\Omega_\Lambda + \Omega_k(1+z)^2 + \Omega_m(1+z)^3}},$$

$$w(\chi,\chi_s) = \frac{H_0^2}{c^2}\,\frac{\mathcal{D}(\chi)\,\mathcal{D}(\chi_s - \chi)}{\mathcal{D}(\chi_s)}\,(1+z),$$

$$\mathcal{D}(\chi) = \frac{cH_0^{-1}}{\sqrt{|\Omega_k|}}\,\mathrm{sinn}\left(\sqrt{|\Omega_k|}\,\chi\right),$$

and where $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$, and "sinn" is defined as sinh if $\Omega_k > 0$, and sin if $\Omega_k < 0$. If $\Omega_k = 0$, the sinn and Ω_k's disappear. The density contrast $\delta \equiv (\rho - \bar\rho)/\bar\rho$. Since $\rho \ge 0$, there exists a minimum value of the convergence:

$$\kappa_{min} = -\frac{3}{2}\,\Omega_m \int_0^{\chi_s} \mathrm{d}\chi\, w(\chi,\chi_s). \qquad (2)$$

The minimum magnification is thus given by $\mu_{min} = 1 + 2\kappa_{min}$.

Now we define (Valageas 2000a)

$$\eta \equiv \frac{\mu - \mu_{min}}{1 - \mu_{min}} = 1 + \frac{\kappa}{|\kappa_{min}|} = \frac{\int_0^{\chi_s} \mathrm{d}\chi\, w(\chi,\chi_s)\,(\rho/\bar\rho)}{\int_0^{\chi_s} \mathrm{d}\chi\, w(\chi,\chi_s)}. \qquad (3)$$

Note that η is the average matter density relative to the global mean, weighted by the gravitational lensing cross section of a unit mass lens along the line of sight to the source. This is the same as the direction-dependent smoothness parameter introduced by Wang (1999) in the weak lensing limit (Wang, in preparation).

The variance of η is given by (Valageas 2000a,b)

$$\xi_\eta = \int_0^{\chi_s} \mathrm{d}\chi\, \left(\frac{w}{F_s}\right)^2 I_\mu(\chi), \qquad (4)$$

with

$$F_s = \int_0^{\chi_s} \mathrm{d}\chi\, w(\chi,\chi_s),$$

$$I_\mu(z) = \pi \int_0^\infty \frac{\mathrm{d}k}{k}\,\frac{\Delta^2(k,z)}{k}\,W^2(\mathcal{D}k\theta_0),$$

where $\Delta^2(k,z) = 4\pi k^3 P(k,z)$, k is the wavenumber, and $P(k,z)$ is the matter power spectrum. The window function $W(\mathcal{D}k\theta_0) = 2J_1(\mathcal{D}k\theta_0)/(\mathcal{D}k\theta_0)$ for smoothing angle θ_0. Here J_1 is the Bessel function of order 1. Using the hierarchical ansatz to model non-linear gravitational clustering (Balian & Schaeffer 1989), Valageas (2000a,b) showed that

$$P(\eta) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}y}{2\pi i \xi_\eta}\, e^{[\eta y - \phi_\eta(y)]/\xi_\eta}, \qquad (5)$$

where $\phi_\eta(y) \simeq \int_0^\infty dx\,(1 - e^{-xy})\,h(x)$. The scaling function $h(x)$ can be obtained from numerical simulations of large scale structure. For $x \ll 1$, $h(x) \propto x^{\omega-2}$ (Valageas 2000a), where ω is the scaling parameter. The uncertainty in Eq.(5) comes primarily from the uncertainty in ω. We found that the scaling function given by Valageas (2000a) leads to large errors in $P(\eta)$ for small ξ_η, making it less useful for calculating $P(\mu)$ at higher redshifts. Although $P(\mu)$ becomes increasingly broad as source redshift increases,

$P(\eta)$ becomes increasingly *narrow*, since the universe becomes more smoothly distributed at high z (Wang 1999).
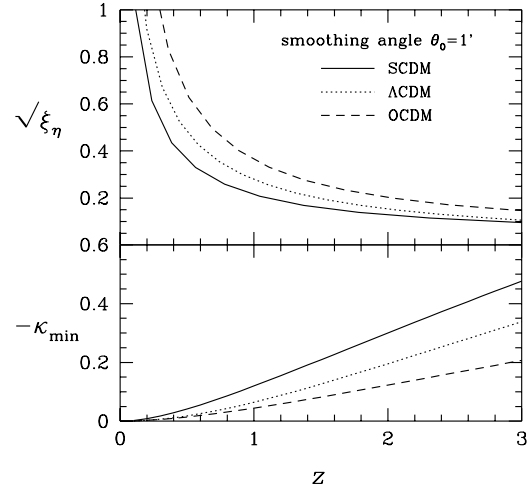

FIG. 1.— $\sqrt{\xi_\eta}$ (for smoothing angle $\theta_0 = 1'$) and $-\kappa_{min}$, for the three cosmological models of Table 1.

3. THE UNIVERSAL PROBABILITY DISTRIBUTION FUNCTION

Munshi & Jain (2000) showed that $P(\eta)$ is independent of the background geometry of the universe, as can be seen from equation (5): since weak lensing contributions are dominated by a narrow range of the matter power spectrum, the scaling function $h(x)$ is independent of cosmological parameters, and hence $P(\eta)$ has no explicit dependence on cosmology (Munshi & Jain 2000). Thus the cosmological dependence of $P(\eta)$ enters entirely through the variance, ξ_η. We can determine the functional form of $P(\eta|\xi_\eta)$ by fitting it to accurate calculations of $P(\mu)$ for *any* cosmological model. The amplification distribution, $P(\mu)$, for arbitrary alternative cosmological models can then be found by computing the appropriate κ_{min} [equation (2)] and ξ_η. Utilizing $\mu = 1 + 2|\kappa_{min}|(\eta - 1)$ we find

$$P(\mu) = \frac{P(\eta|\xi_\eta)}{2|\kappa_{min}|}. \qquad (6)$$

We call $P(\eta)$ the universal probability distribution function (UPDF), as this one-parameter family of solutions underlies all weak lensing magnification PDFs for *all* cosmologies, at *all* redshifts. We expect our results to be valid in the weak lensing limit, for $\kappa \lesssim 0.2$. In particular, the PDFs derived using our formulae are not expected to have accurate high magnification tails.

Figure 1 shows $\sqrt{\xi_\eta}$ and $-\kappa_{min}$ computed using equations (4) and (2), for the three cosmological models from Table 1. For illustration, we give accurate fitting formulae in Table 2 for the curves in Figure 1.

Table 2
Fitting formulae for curves in Fig.1:
$\sqrt{\xi_\eta} = \sum_{i=0}^3 a_i(5z)^{-i}$, $-\kappa_{min} = \sum_{i=0}^3 a_i(z/5)^i$

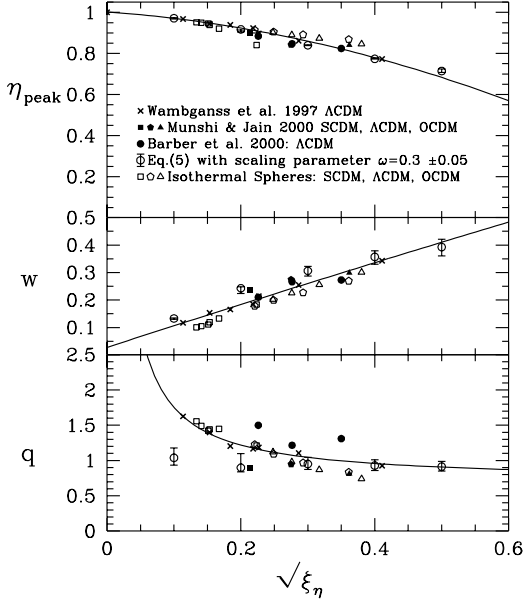|	a_0	a_1	a_2	a_3	a_0	a_1	a_2	a_3
SCDM	.032	.986	$-.452$	.114	$-.025$	.667	.482	$-.337$
ΛCDM	.021	1.384	$-.642$	.147	$-.015$	.280	.766	$-.426$
OCDM	.032	1.761	$-.648$	.146	$-.004$	.121	.703	$-.538$

FIG. 2.— The dependence of the three UPDF fitting parameters, as functions of $\sqrt{\xi_\eta}$. Data for a variety of different models, over a range of redshift, is shown. In addition, best fit curves to the Wambsganss et al. points are superposed (see equation 8).

We extract the UPDF, $P(\eta)$, from ray-tracing within the large scale structure simulations of Wambsganss et al. (1997). We then fit the UPDF to the stretched Gaussian (Wang 1999):

$$P(\eta|\xi_\eta) = C_{norm} \, \exp\left[-\left(\frac{\eta - \eta_{peak}}{w \, \eta^q}\right)^2\right], \qquad (7)$$

where C_{norm}, η_{peak}, w, and q depend solely on ξ_η and are independent of η. Note that although eq.(7) accurately describes the shape of the UPDF in the range of η which is relevant to weak lensing, one must impose self-consistency by restricting $\eta \leq \eta_{max}$, with η_{max} chosen such that eq.(7) gives the correct ξ_η. Typically, $\eta_{max} \sim 3-7$ for $1 \leq z \leq 3$ in a ΛCDM model. We find that $\langle\eta\rangle = 1$ (hence $\langle\mu\rangle = 1$) for $\eta \leq \eta_{max}$. This is because η_{max} is sufficiently large so that the contribution of the high η tail to the mean is negligible. $C_{norm}(\xi_\eta)$ is a normalization constant, chosen so that $\int_0^{\eta_{max}} P(\eta)\,d\eta = 1$. Figure 2 shows η_{peak}, w, and q as functions of $\sqrt{\xi_\eta}$. The points denoted by crosses are extracted from the numerical $P(\mu|z)$ by Wambsganss et al. (1997) for a ΛCDM model with $\Omega_m = 0.4$, $\Omega_\Lambda = 0.6$; the solid curves are (χ^2 minimizing) best fits to the crosses:

$$\eta_{peak}(\xi_\eta) = 1.002 - 1.145\left(\frac{\sqrt{\xi_\eta}}{5}\right) - 20.427\left(\frac{\sqrt{\xi_\eta}}{5}\right)^2,$$

$$w(\xi_\eta) = .028 + 3.952\left(\frac{\sqrt{\xi_\eta}}{5}\right) - 1.262\left(\frac{\sqrt{\xi_\eta}}{5}\right)^2, \quad (8)$$

$$q(\xi_\eta) = .702 + .509\left(\frac{1}{5\sqrt{\xi_\eta}}\right) + .008\left(\frac{1}{5\sqrt{\xi_\eta}}\right)^2.$$

The parameter $\eta_{peak}(\xi_\eta)$ indicates the average smoothness of a universe; it increases with decreasing ξ_η (i.e., increasing z) and approaches $\eta_{peak}(\xi_\eta) = 1$ for $\xi_\eta \to 0$. The

parameter $w(\xi_\eta)$ indicates the width of the distribution in the smoothness parameter η; it decreases with decreasing ξ_η (i.e., increasing z). The ξ_η dependences of $\eta_{peak}(\xi_\eta)$ and $w(\xi_\eta)$ are as expected because as we look back to earlier times, lines of sight sample more of the universe, and the universe becomes smoother on average. The parameter $q(\xi_\eta)$ indicates the deviation of $P(\eta|\xi_\eta)$ from Gaussianity (which corresponds to $q = 0$).

4. COMPARISON WITH OTHER PUBLISHED RESULTS

The universal probability distribution function, $P(\eta)$, encapsulated in eqs. (7) and (8), can be used to determine the magnification probability distribution, $P(\mu)$, for arbitrary cosmological models at arbitrary redshifts. For each parameter and redshift, the single free parameter ξ_η determines the full probability distribution.
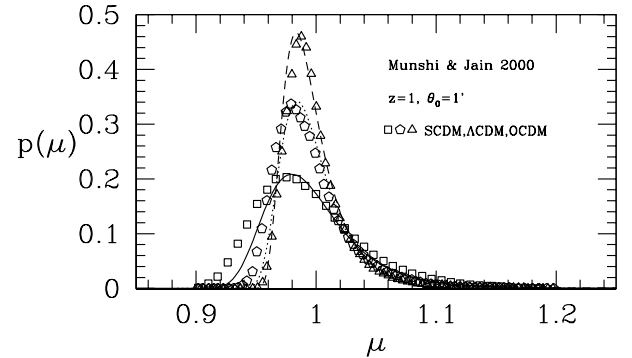

FIG. 3.— The amplification probability distribution, $P(\mu)$, derived from ray-tracing simulations by Munshi & Jain (2000) (open symbols) for smoothing angle $\theta_0 = 1'$, source redshift $z_s = 1$, and the three cosmological models of Table 1, together with $P(\mu)$ computed using our UPDF, with κ_{min} and ξ_η computed using equations (2) and (4).
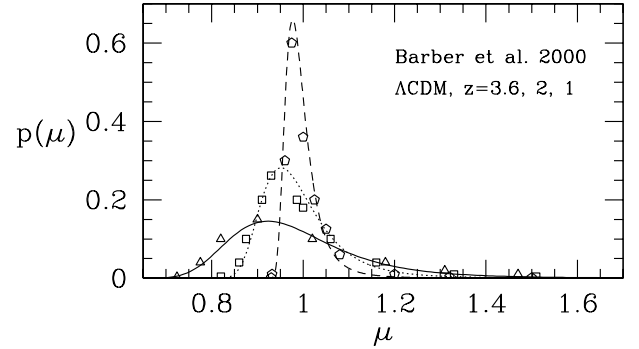

FIG. 4.— The $P(\mu)$ from three-dimensional shear matrix calculations of N-body simulations by Barber et al. (2000) (circles) for a ΛCDM model with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ at source redshifts $z_s = 3.6, 2, 1$ (peaking from left to right), together with $P(\mu)$ computed using our UPDF, with κ_{min} computed using equation (2) and ξ_η inferred from Table 4 of Barber et al. (2000).

Figure 3 shows the $P(\mu)$ from ray-tracing simulations by Munshi & Jain (2000) for smoothing angle $\theta_0 = 1'$, source redshift $z_s = 1$, and three cosmological models from Table 1, together with $P(\mu)$ computed using our UPDF for the κ_{min} and ξ_η computed using equations (2) and (4).

Figure 4 shows the $P(\mu)$ from three-dimensional shear matrix calculations of N-body simulations by Barber et

al. (2000) for a ΛCDM model with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ at source redshifts $z_s = 1$, 2, 3.6, together with $P(\mu)$ computed using our UPDF, with κ_{min} computed using equation (2) and ξ_η inferred from Table 4 of Barber et al. (2000) [equation (4) was not used to compute ξ_η due to our lack of knowledge of the smoothing angle θ_0 that corresponds to their results].

Our UPDF gives $P(\mu)$ in excellent agreement with N-body calculations. To make this more apparent, we have extracted $P(\eta)$ from the $P(\mu)$ obtained via N-body calculations by Munshi & Jain (2000) and Barber et al. (2000), and fitted them to the functional form of equation (7). The resultant coefficients are plotted in Figure 2. There is very good agreement in the peak location $\eta_{peak}(\xi_\eta)$ and width indicator $w(\xi_\eta)$, but larger scatter in the non-Gaussianity indicator $q(\xi_\eta)$ extracted from Munshi & Jain (2000) and Barber et al. (2000). The latter may arise partly due to the fact that in both cases we poorly resolve the non-Gaussian tails, which are crucial to determining accurate values of q. In addition, the weak lensing condition breaks down for the high μ tails, which could be significant for small ξ_η. Also plotted in Figure 2 are the coefficients extracted from fitting the analytically computed $P(\eta)$ [see Eq.(5)], following Munshi & Jain (2000), for the scaling parameter $\omega = 0.3 \pm 0.05$. These $P(\eta)$ have not been tested for $z > 1$ (i.e., for small ξ_η), although the deviations are expected to be small, since $P(\eta)$ peaks close to $\eta = 1$ at small ξ_η [see equation (7)].

Figure 2 also shows the $P(\eta)$ coefficients extracted from ray-tracing of randomly placed singular isothermal spheres (SIS), following the prescription of Holz & Wald (1998), for the three cosmological models of Table 1; these are in good agreement with the fitted coefficients from Wambsganss et al. (1997). We find that improved statistics leads to better agreement between the $q(\xi_\eta)$ from our ray-tracing of randomly placed mass distributions and that from Wambsganss et al. (1997), while having much less impact on $\eta_{peak}(\xi_\eta)$ and $w(\xi_\eta)$. This is as expected, since improved statistics fills out the non-Gaussian tails of the $P(\mu)$.

5. SUMMARY AND DISCUSSION

We have derived a simple and accurate method to compute the weak lensing magnification distribution, $P(\mu)$, for standard candles placed at any redshift in arbitrary cosmological models. We use a universal probability distribution function (UPDF), $P(\eta|\xi_\eta)$, which is independent of cosmological model; the dependence on cosmology entering only through the variance, ξ_η, of the reduced convergence, η. The UPDF is fit accurately by a 3-parameter stretched Gaussian distribution [eq. (7)]. We give polynomial fitting formulae [eq. (8)] for the three parameters $\eta_{peak}(\xi_\eta)$ (average smoothness), $w(\xi_\eta)$ (smoothness variation), and $q(\xi_\eta)$ (non-Gaussianity), which we normalize to the N-body simulations of Wambsganss et al. (1997). The magnification PDF, $P(\mu, z)$, can then be determined from the UPDF using equation (6). We expect our results to be valid in the weak lensing limit, for $\kappa \lesssim 0.2$. The extension of our method to high magnifications will be presented elsewhere.

To test the robustness of this method, we have compared our results against three alternate independent methods (see Fig. 2). We find excellent agreement with: (1) the N-body calculations of Munshi & Jain (2000) and Barber et al. (2000), with some scatter in the non-Gaussianity indicator $q(\xi_\eta)$, which is consistent with the limited statistics at low z (i.e., large ξ_η) of these N-body results and the breaking down of the weak lensing condition at high μ, (2) the analytical calculation [see equation (5)] following Munshi & Jain (2000), where the latter has been verified by ray tracing experiments, and (3) the ray-shooting of randomly placed SIS mass distributions, following the prescription of Holz & Wald (1998).

We expect these simple, universal forms for the weak lensing distribution to be useful in addressing high redshift data, and in particular, in the analysis of results from future supernova surveys.

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